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Two-point correlations of the Gaussian symplectic ensemble from periodic orbits

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Abstract. We determine the asymptotics of the two-point correlation function for quantum systems with half-integer spin which show chaotic behaviour in the classical limit using a method introduced by Bogomolny and Keating [Phys. Rev. Lett. **77** (1996) 1472–1475]. For time-reversal invariant systems we obtain the leading terms of the two-point correlation function of the Gaussian symplectic ensemble. Special attention has to be paid to the rôle of Kramers' degeneracy.

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Understanding correlations of energy levels of quantum mechanical systems whose classical limit exhibits chaotic motion is one of the major topics in quantum chaos. The bridge between quantum mechanics and classical mechanics is provided by the Gutzwiller trace formula [1] which relates the quantum mechanical density of states $d(E) = \sum_n \delta(E - E_n)$ to a sum over periodic orbits of the corresponding classical system,

$$d(E) \sim \bar{d}(E) + \frac{1}{2\pi\hbar} \sum_{\gamma} \sum_{k \in \mathbb{Z} \setminus \{0\}} \mathcal{A}_{\gamma k} T_{\gamma} e^{\frac{i}{\hbar} k S_{\gamma}(E)}, \quad \hbar \rightarrow 0, \quad (1)$$

where $\bar{d}(E)$ denotes the mean spectral density (which, by Weyl's law, is of order \hbar^{-f} for systems with f degrees of freedom), and the sum extends over all primitive periodic orbits γ and their repetitions, formally including negative ones. $S_{\gamma}(E) = \oint_{\gamma} \vec{p} d\vec{x}$ denotes the classical action, T_{γ} is the (primitive) period, $T_{\gamma} = dS_{\gamma}(E)/dE$, and the amplitude $\mathcal{A}_{\gamma k}$ involves topological and stability properties. The conjecture of Bohigas, Giannoni and Schmit (BGS) [2] states that for classically chaotic systems, generically, the statistics of energy levels can be modeled by the average behaviour of ensembles of random matrices. In the case of systems without spin the relevant ensembles are the Gaussian orthogonal and the Gaussian unitary ensemble (GOE/GUE) depending on whether the system does or does not possess an antiunitary symmetry like time-reversal, see, e.g. [3]. In the case of time-reversal invariant systems with half-integer spin one also has to deal with the Gaussian symplectic ensemble (GSE).

The main result in understanding eigenvalue correlations in terms of the underlying classical dynamics is due to Berry [4]. He used the so-called diagonal approximation and the Hannay-Ozorio de Almeida sum rule [5], see also [6], to determine the asymptotics of the spectral form factor, which is the Fourier transform of the two-point correlation function $R_2(s)$, see eq. (7) below. This treatment has recently been generalized to the case with half-integer spin [7] using an analogue of the Gutzwiller trace formula which includes an additional factor due to spin precession [8, 9].

In the case of the GOE and the GUE Bogomolny and Keating [10], see also [11, 12], developed a method for the semiclassical evaluation of $R_2(s)$ which yields an additional term as compared to the diagonal approximation of the form factor. More precisely, their method yields the leading non-oscillatory and the leading oscillatory term of $R_2(s)$ as $s \rightarrow \infty$, whereas the diagonal approximation of the form factor corresponds to the leading non-oscillatory term. Recently Haake [3] proposed a method to adapt this result to the case of the GSE. But, surprisingly, although he obtained two terms of the large s asymptotics of $R_2(s)$, the method failed to reproduce the leading term. The aim of this Letter is to present a slightly different approach to systems with half-integer spin which correctly yields both the leading non-oscillatory and the leading oscillatory term. Note that [10] also includes a remark on GSE asymptotics which, however, is not based on semiclassics with spin but on a theorem in random matrix theory and, therefore, is not to be confused with the problem addressed here.

The general method of [10] consists of three main steps. Starting from the

observation that trace formulae lead to accurate semiclassical quantization conditions Bogomolny and Keating propose to base the semiclassical analysis of spectral correlations on such an approximate spectrum. In the course of the calculations they, secondly, employ the diagonal approximation as introduced in [5, 4]. Finally, they make use of the assumption that the oscillating part of the integrated spectral density (i.e. the contribution of periodic orbits) behaves like a Gaussian random variable. Here we will only briefly sketch the necessary changes to the method of Bogomolny and Keating in order to take care of the situation with half-integer spin. For the general formalism we refer to [10, 11, 12, 3]. We will also rely heavily on results of [7].

In order to obtain a simple but efficient semiclassical quantization condition, we first integrate (1) over the energy E which yields a trace formula for the spectral staircase function $N(E)$. Taking into account only orbits up to a time T , which below will be chosen of the order of Heisenberg time $T_H = 2\pi\hbar\bar{d}(E)$, one obtains a truncated spectral staircase function $N_T(E)$, and the semiclassical eigenvalues $E_n(T)$ can be determined from the condition [13, 14]

$$N_T(E_n(T)) \stackrel{!}{=} n + \frac{1}{2}. \quad (2)$$

The trace formula (1) can easily be integrated if there is a one-to-one correspondence between orbits at different energies (i.e. no bifurcations occur when varying E) and from successive integration by parts we see that in leading order in \hbar it is sufficient to integrate the exponential, i.e

$$N_T(E) \sim \bar{N}(E) + \sum_{\gamma} \sum_{\substack{k \in \mathbb{Z} \setminus \{0\} \\ |k|T_{\gamma} \leq T}} \frac{1}{2\pi i k} \mathcal{A}_{\gamma k} e^{\frac{i}{\hbar} k S_{\gamma}(E)}, \quad \hbar \rightarrow 0, \quad (3)$$

where the periodic orbit sum will later be denoted by $N_T^{\text{fluc}}(E)$. At this point it is important to take care of Kramers' degeneracy. If the quantum system, with Hamiltonian \hat{H} , has half-integer spin and is invariant under time-reversal, i.e. $[\hat{H}, \hat{T}] = 0$ with $\hat{T} = i\sigma_y \hat{K}$, where \hat{K} is the operator of complex conjugation, then each energy eigenvalue has at least multiplicity two. One could now attempt to first calculate the correlations for the degenerate spectrum and relate the result to the correlations of the non-degenerate spectrum, cf. [3]. This strategy is successful for the form factor [7]. However, since the truncated spectral staircase function $N_T(E)$ fails to reproduce sharp steps of size two, the quantization condition (2) will not yield degenerate eigenvalues but two distinct eigenvalues which both have an additional error. Therefore, we instead take Kramers' degeneracy into account at this point by imposing the modified quantization condition

$$N_T(E_n(T)) \stackrel{!}{=} 2n + 1 \quad (4)$$

which produces a semiclassical spectrum $\{E_n(T)\}$ with Kramers' degeneracy already removed. Note that this semiclassical spectrum has mean density $\bar{d}(E)/2$, and the corresponding Heisenberg time is $T_H = \pi\hbar\bar{d}(E)$. Using the Poisson summation formula,

the density of states $\tilde{d}(E)$ of the semiclassical spectrum can be written as

$$\tilde{d}(E) := \sum_n \delta(E - E_n(T)) = \frac{1}{2} d_T(E) \sum_{\nu \in \mathbb{Z}} (-1)^\nu e^{i\pi \nu N_T(E)}, \quad (5)$$

where $d_T(E) = dN_T(E)/dT$, see (3). Before we can compare spectral correlations with results from random matrix theory (RMT) the spectrum has to be unfolded, i.e. the eigenenergies are rescaled such that their mean separation is one. To this end consider the spectral interval $I = I(E, \hbar) := [E - \hbar\omega, E + \hbar\omega]$ which contains N_I levels. In the semiclassical limit this number can be estimated by $N_I \sim 2\hbar\omega\bar{d}/2$, where $\bar{d} = \bar{d}(E)$, i.e. as $\hbar \rightarrow 0$ the length of the interval shrinks to zero but the number of eigenvalues contained in $I(E, \hbar)$ goes to infinity, cf. [7]. Defining the unfolded eigenvalues by $x_n(T) := E_n(T)\bar{d}/2$, the density of states $D_T(x)$, $x = E\bar{d}/2$, of the unfolded spectrum $\{x_n(T)\}$ reads

$$D_T(x) := \sum_n \delta(x - x_n(T)) = \frac{2}{\bar{d}} \tilde{d}_T(E). \quad (6)$$

The semiclassical two-point correlation function is defined by

$$R_2(s, I) := \frac{1}{\hbar\omega\bar{d}} \int_{x-\hbar\omega\bar{d}/2}^{x+\hbar\omega\bar{d}/2} D_T\left(x' + \frac{s}{2}\right) D_T\left(x' - \frac{s}{2}\right) dx' - 1. \quad (7)$$

From the BGS-conjecture we expect that in the semiclassical limit $R_2(s, I)$ converges weakly to the random matrix result ($R_2^{\text{GSE}}(s)$ in the case considered here), i.e.

$$\lim_{\hbar \rightarrow 0} \int_{\mathbb{R}} R_2(s, I) \phi(s) ds = \int_{\mathbb{R}} R_2^{\text{GSE}}(s) \phi(s) ds \quad (8)$$

for any smooth test function $\phi \in \mathcal{S}(\mathbb{R})$. We only aim at providing a periodic orbit theory for this relation in the combined limit

$$s \rightarrow \infty, \quad \bar{d} \rightarrow \infty \quad \text{and} \quad s/\bar{d} \rightarrow 0 \quad (9)$$

which will allow expansions in s/\bar{d} . Here $\bar{d} \rightarrow \infty$ is a consequence of the semiclassical limit and Weyl's law. The asymptotics of the GSE-result reads (see, e.g. [15])

$$R_2^{\text{GSE}}(s) \sim \frac{\pi \cos(2\pi s)}{2} - \frac{1 + \frac{\pi}{2} \sin(2\pi s)}{(2\pi s)^2}, \quad s \rightarrow \infty. \quad (10)$$

Substituting (6) and (5) into (7) results in

$$R_2(s, I) = \frac{1}{\bar{d}^2} \left\langle d_T\left(E' + \frac{s}{\bar{d}}\right) d_T\left(E' - \frac{s}{\bar{d}}\right) \sum_{\nu, \nu' \in \mathbb{Z}} (-1)^{\nu-\nu'} e^{i\pi(\nu N_T(E' + \frac{s}{\bar{d}}) - \nu' N_T(E' - \frac{s}{\bar{d}}))} \right\rangle_{E'} - 1, \quad (11)$$

where the brackets denote an average over $I(E, \hbar)$, i.e. $\langle \dots \rangle_{E'} = \frac{1}{2\hbar\omega} \int_{E-\hbar\omega}^{E+\hbar\omega} \dots dE'$. By a stationary phase argument one easily sees that the terms with $\nu \neq \nu'$ are of relative order $O(1/\bar{d})$ in the desired limit (9), i.e. we have

$$R_2(s, I) \sim \sum_{\nu \in \mathbb{Z}} r_\nu(s, I) \quad (12)$$

with

$$r_\nu(s, I) := \frac{1}{\bar{d}^2} \left\langle d_T\left(E' + \frac{s}{\bar{d}}\right) d_T\left(E' - \frac{s}{\bar{d}}\right) e^{i\pi\nu(N_T(E' + \frac{s}{\bar{d}}) - N_T(E' - \frac{s}{\bar{d}}))} \right\rangle_{E'} - \delta_{\nu 0}. \quad (13)$$

The evaluation of $r_0(s, I)$ is straight forward and will not be shown here. The result corresponds to the usual diagonal approximation of the form factor (cf. [10, 3]) and therefore in the present situation reads [7]

$$r_0(s, I) \approx -\frac{1}{(2\pi s)^2}, \quad (14)$$

which is the leading non-oscillating contribution of $R_2^{\text{GSE}}(s)$ as $s \rightarrow \infty$ (10). Here ‘ \approx ’ indicates that (14) is not just an asymptotic relation but we have also used the diagonal approximation which is assumed to be valid in the combined limit (9). Further conditions needed to arrive at (14) are hyperbolicity of the translational dynamics and the mixing property of the skew product of translational and spin dynamics, see [7] for details. We remark that the last condition can be weakened to ergodicity; this result will be presented elsewhere [16]. Introducing an auxiliary variable s' , the contributions $r_\nu(s, I)$, $\nu \neq 0$, can be written as derivatives,

$$r_\nu(s, I) \sim \frac{1}{(i\pi\nu)^2} \frac{\partial^2}{\partial s \partial s'} e^{i\pi\nu(s+s')} \Phi_\nu(s, s') \Big|_{s'=s}, \quad (15)$$

where we have expanded the smooth part $\tilde{N}(E)$ of $N_T(E)$ in powers of s/\bar{d} . The functions $\Phi_\nu(s, s')$ are then defined by

$$\Phi_\nu(s, s') := \left\langle e^{i\pi\nu \left(N_T^{\text{fluc}}(E' + \frac{s}{d}) - N_T^{\text{fluc}}(E' - \frac{s'}{d}) \right)} \right\rangle_{E'}. \quad (16)$$

The next step lies at the heart of the method of [10]. Assuming that the exponent of (16) behaves like a Gaussian random variable $G(E')$ with zero mean we can use the identity $\langle \exp(iG(E')) \rangle_{E'} = \exp(\langle -G^2(E')/2 \rangle_{E'})$ and subsequently evaluate the exponent in diagonal approximation. This assumption is favoured by a well established conjecture on global eigenvalue correlations [17]. Employing an expansion in s/\bar{d} the difference in the exponent of (16) reads

$$N_T^{\text{fluc}}(E' + \frac{s}{d}) - N_T^{\text{fluc}}(E' - \frac{s'}{d}) \sim \sum_{\gamma} \sum_{\substack{k \in \mathbb{Z} \setminus \{0\} \\ |k|T_\gamma \leq T}} \frac{\mathcal{A}_{\gamma k}}{2\pi i k} e^{\frac{i}{\hbar} k S_\gamma(E')} \left(e^{\frac{i}{\hbar} k T_\gamma(E') \frac{s}{d}} - e^{-\frac{i}{\hbar} k T_\gamma(E') \frac{s'}{d}} \right). \quad (17)$$

For the square of this expression we again make use of the diagonal approximation, which was already needed to evaluate $r_0(s, I)$, i.e. we only keep contributions of orbits with like actions,

$$\begin{aligned} & \left\langle \left(N_T^{\text{fluc}}(E' + \frac{s}{d}) - N_T^{\text{fluc}}(E' - \frac{s'}{d}) \right)^2 \right\rangle_{E'} \approx \\ & \left\langle \sum_{\gamma} \sum_{\substack{k \in \mathbb{Z} \setminus \{0\} \\ |k|T_\gamma \leq T}} \frac{g}{(2\pi k)^2} |\mathcal{A}_{\gamma k}|^2 2 \left(1 - \cos \left(\frac{k}{\hbar} T_\gamma(E') \frac{s+s'}{\bar{d}} \right) \right) \right\rangle_{E'}, \quad (18) \end{aligned}$$

where the generic multiplicity of periodic orbits, see, e.g., [7], will be set to $g = 2$, since we are dealing with time-reversal invariant systems. The remaining sum over periodic orbits can be evaluated with the sum rule of [7], which, essentially, is the Hannay-Ozorio

de Almeida sum rule [5], additionally taking into account the spin contribution. This yields

$$\begin{aligned} \left\langle \left(N_T^{\text{fluc}}(E' + \frac{s}{d}) - N_T^{\text{fluc}}(E' - \frac{s'}{d}) \right)^2 \right\rangle_{E'} &\approx \frac{g}{\pi^2} \int_0^T \frac{1 - \cos\left(\frac{s+s'}{\hbar d} T'\right)}{T'} dT' \\ &\sim \frac{g}{\pi^2} \log\left(\frac{s+s'}{\hbar d} T\right). \end{aligned} \quad (19)$$

At this point a short remark is in order. The final result for $r_\nu(s, I)$, $\nu \neq 0$, will still include the cut-off time T . It has been argued [10, 3] that T has to be chosen of the order of Heisenberg time, i.e. $T = C\pi\hbar\bar{d}$, where the constant C has to be determined by comparing to the asymptotics of the RMT-result. However, the result for C will depend sensitively on the second term in the asymptotic expansion of the cosine-integral in (19). Terms of the same order could also arise from non-leading contributions to the sum rule, which, unfortunately, are unknown. We are thus unable to give the correct sub-leading term in the asymptotic expansion (19). Therefore we conclude that from the above considerations we can only obtain the s -dependence of $r_\nu(s, I)$, $\nu \neq 0$, but must refrain from any discussion about the cut-off time T . Putting together (15), (16), (19) and the Gaussian ansatz we first observe that contributions with $|\nu| > 1$ decay rapidly. The leading correction to (14) is hence given by $r_1(s, I) + r_{-1}(s, I)$, which yields a term proportional to $\cos(2\pi s)/s$. This result is consistent with the leading oscillatory term of $R_2^{\text{GSE}}(s)$, cf. (10).

Summarizing, the method of Bogomolny and Keating [10] has been applied to time-reversal invariant systems with half-integer spin. As in the previously studied cases without spin it correctly reproduces the leading non-oscillatory term and the s -dependence of the leading oscillatory term of the two-point correlation function as $s \rightarrow \infty$. Although giving further semiclassical evidence towards the BGS-conjecture open questions, as, e.g., a consistent determination of the correct cut-off time T , remain.

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